Algebraic Geometry Example Sheet 2: Lent 2025

Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `hk439@cam.ac.uk`. In all questions, k is an algebraically closed field of characteristic 0.

1. Show that the set of algebraic subsets of $\mathbb{P}^n$ are the closed sets of a topology on $\mathbb{P}^n$.

2. Given distinct points $P_0, \ldots, P_{n+1}$ in $\mathbb{P}^n$, no $(n+1)$ of which are contained in a hyperplane, show that homogeneous coordinates may be chosen on $\mathbb{P}^n$ so that $P_0 = (1 : 0 : \ldots : 0), \cdots, P_n = (0 : \ldots : 0 : 1), P_{n+1} = (1 : 1 : \ldots : 1)$. [This generalizes to arbitrary n a result you are familiar with from complex analysis when $n = 1$.]

3. Given hyperplanes $H_0, \ldots, H_n$ of $\mathbb{P}^n$ which satisfy $H_0 \cap \cdots \cap H_n = \emptyset$, show that homogeneous coordinates $x_0, \ldots, x_n$ can be chosen so that each H_i is the locus $x_i = 0$.

4. Let V be a hypersurface in $\mathbb{P}^n$ and L a projective line in $\mathbb{P}^n$. Show that V and L intersect in a non-empty set. [A projective line in $\mathbb{P}^n$ is a subvariety defined by $n - 1$ linearly independent homogeneous linear equations.]

5. Write down the projective closures of the following affine plane curves and compute their intersections with the three coordinate lines $Z(x), Z(y), Z(z)$ in $\mathbb{P}^2$.

$$C_1 : xy = x^6 + y^6,$$

$$C_2 : x^3 = y^2 + x^4 + y^4.$$

6. The <u>Segre surface</u> $\Sigma_{1,1} \subset \mathbb{P}^3$ is given by $Z(x_0x_3 - x_1x_2)$. Find a pair of disjoint lines contained in $\Sigma_{1,1}$. Find a pair of intersecting lines contained in $\Sigma_{1,1}$.

7. Consider the affine twisted cubic $V = \{(t, t^2, t^3) \ t \in k\} \subset \mathbb{A}^3$. Observe that $V = Z(x_2 - x_1^2, x_3 - x_1^3)$ and V is irreducible. Show that $Z(x_2x_0 - x_1^2, x_3x_0^2 - x_1^3) \subset \mathbb{P}^3$ is not irreducible. Compute generators for the ideal of the projective closure of V in $\mathbb{P}^3$.

8. Consider the cubic surface $S \subset \mathbb{P}^3$ given by $Z(x_0^3 - x_1^3 + x_2^3 - x_3^3)$. Find a line ℓ contained on this surface. Find a hyperplane $P \subset \mathbb{P}^3$ which contains your line with $S \cap P$ a union of three lines.

9. The <u>Cremona involution</u> on $\mathbb{P}^2$ is the map $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ defined by

$$(x_0 : x_1 : x_2) \mapsto (x_1x_2 : x_0x_2 : x_0x_1).$$

Show this is a (birational) involution. Let ℓ be the line $Z(x_0 + x_1 + x_2)$ and let $U \subset \mathbb{P}^2$ be an open set in the domain of φ. Calculate the ideal of the Zariski closure of $\varphi(U \cap \ell)$.

10. Let $Q \subset \mathbb{P}^{n+1}$ be an irreducible quadric hypersurface. Prove that Q is birational to $\mathbb{P}^n$ and use this to calculate the function field of Q. [Hint: think about how we parametrized the circle in lecture 1].